

Geometries from Young diagrams

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ABSTRACT: Type IIB string theory on spacetimes that are asymptotically $\text{AdS}_5 \times \text{S}^5$ can be defined using four dimensional $\mathcal{N} = 4$ super Yang-Mills theory. Six of the dimensions of the string theory are holographically reconstructed in the Yang-Mills theory. In this article we study how these dimensions and local physics in these dimensions emerge. We reorganize the dynamics of the $\frac{1}{2}$ BPS sector of the field theory by rewriting it in terms of Schur polynomials. The Young diagram labeling of these polynomials can be viewed as a book keeping device which summarizes how the operator is constructed. We show that aspects of the geometry of the extra holographic dimensions are captured very naturally by the Young diagram. Gravitons which are localized at a specific position in the bulk correspond to boxes added at a specific location on the Young diagram.

KEYWORDS: Brane Dynamics in Gauge Theories, AdS-CFT Correspondence.

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1. Introduction

We do not yet know what string theory is. AdS/CFT [1] seems to provide us with a definition of string theory on negatively curved spaces; for example, type IIB string theory on spacetimes that are asymptotically $\text{AdS}_5 \times \text{S}^5$ is given by four dimensional $\mathcal{N} = 4$ super Yang-Mills theory (see [2] for an explicit demonstration of this in the $\frac{1}{2}$ -BPS sector). There are many aspects of this definition that need to be clarified. The string theory is a 10 dimensional theory; apparently, there are an additional 6 dimensions that are holographically reconstructed in the Yang-Mills theory. Understanding how these dimensions emerge and further, how *local physics* in these dimensions emerge is a fascinating problem. In this article, we explore this issue.

The dynamical content of any quantum field theory is captured in its correlation functions. The extra holographic dimensions together with their local dynamics should be coded into the correlation functions of $\mathcal{N} = 4$ super Yang-Mills theory. See [3] for detailed examples of what information is coded and how its coded. According to the AdS/CFT correspondence, we expect a classical geometry to emerge in the strong 't Hooft coupling limit of the field theory. In this article, we focus mainly on correlation functions of $\frac{1}{2}$ BPS operators because these correlators are protected [4]. Thus, we will compute these correlation functions at weak coupling with confidence that these free field results can be extrapolated, with no change, to strong coupling.

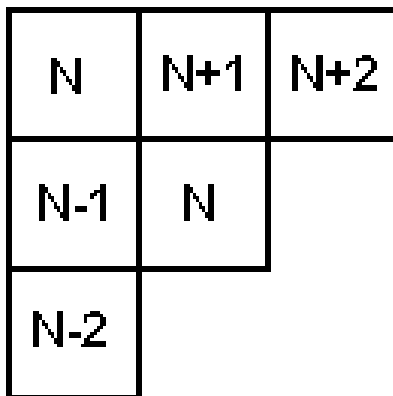


Figure 1: A Young diagram with the weight of each box displayed.

Our approach entails performing a reorganization of the dynamics of the field theory. The $\frac{1}{2}$ BPS sector of the theory is captured by the singlet dynamics of a single holomorphic matrix [5, 6]. The reorganization we employ is achieved by a change of variables: we will use Schur polynomials which provide a complete description of the $\frac{1}{2}$ BPS sector [5].¹ This reorganization will allow a significant simplification of the computation of correlators for two reasons. First, the Schur polynomials satisfy a nice product rule that can be used to collapse any product of Schur polynomials into a sum of Schur polynomials. Second, the two point functions of Schur polynomials, in the free field theory limit, are known exactly [5]. Thus, this reorganization provides a considerable simplification of correlator calculations. The label of the Schur polynomial, a Young diagram, plays a central role.

We will start with a quick review of Young diagrams. We have two goals in mind: (i) to recall the definition of the hooks and the weights of a Young diagram and (ii) to stress that the hooks and weights encode the specific way in which a tensor has been symmetrized or antisymmetrized.

Young diagrams are a useful way to label the irreducible representations of $SU(N)$.² A tensor transforming in an irreducible representation of $SU(N)$ has a definite symmetry under permutations of its indices. It is this symmetry that the Young diagram records. A Young diagram with a single column containing n -boxes corresponds to a completely antisymmetric tensor with n indices. A Young diagram with a single row containing m -boxes corresponds to a completely symmetric tensor with m indices. To compute the dimension of an $SU(N)$ irreducible representation corresponding to a particular Young diagram, we need to define the weight³ and the hook of each box in the Young diagram. A box in row i and column j has a weight equal to $N - i + j$. An example of a Young diagram with the weights filled in is given in figure 1.

¹They are simply an alternative basis to the trace basis.

²A Young diagram with n boxes also labels an irreducible representation of the symmetric group S_n . We will also exploit this connection in what follows.

³These are not the Dynkin weights.

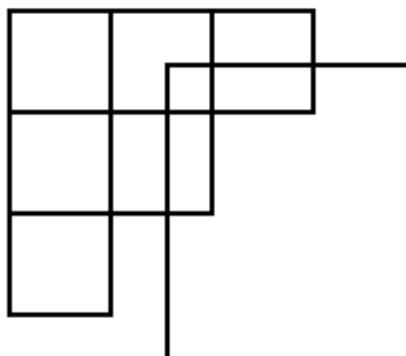


Figure 2: The hook shown is associated with the box in which the corner of the elbow lies.

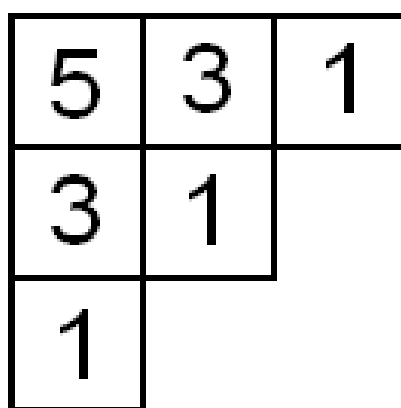


Figure 3: A Young diagram with the hook length of each box displayed.

To obtain the hook associated to a given box, draw a line starting from the given box towards the bottom of the page until you exit the Young diagram, and another line starting from the same box towards the right until you again exit the diagram. These two lines form an elbow - what we call the hook. An example of a hook is given in figure 3.

The hook length for the given box is obtained by counting the number of boxes the elbow belonging to the box passes through. A Young diagram with the hook lengths filled in is given in figure 3.

The dimension of the $SU(N)$ irreducible representation corresponding to this Young diagram is given by the product of the weights divided by the product of the hooks. For the example we are considering

$$d_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}} = \frac{N^2(N^2 - 1)(N^2 - 4)}{5 \cdot 3 \cdot 3}.$$

The hooks and the weights provide an efficient way to encode the combinatorics of tensors with a definite symmetry under swapping indices. For example, the totally antisymmetric tensor with three indices, gives a non-zero result only if all indices take distinct values.

Thus, the first index can take any one of N values, the second index any one of $N - 1$ values and the third index any one of $N - 2$ values. These are exactly the value of the weights of the corresponding Young diagram. The division by the hooks corrects for the fact that not all elements of this tensor are independent - swapping any two indices only costs a sign.

Recently it has become clear that Young diagrams also provide a useful labeling of the invariant variables of matrix models. The first such example was given in [5], where it was shown that Schur polynomials built using a single complex scalar Z provide a convenient parametrization of the $\frac{1}{2}$ BPS sector of the theory. A particularly nice property of the Schur polynomials

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) Z_{i_{\sigma(1)}}^{i_1} \cdots Z_{i_{\sigma(n)}}^{i_n},$$

is that they have a diagonal two point function

$$\langle \chi_R(Z)^\dagger \chi_S(Z) \rangle = f_R \delta_{RS}. \tag{1.1}$$

If R is a Young diagram with n boxes, then the Schur polynomial is built using products of traces of Higgs fields Z such that each term is a product of n Z s. The factor f_R is equal to the product of the weights of R . In addition to this, the Schur polynomials also satisfy a product rule called the Littlewood-Richardson rule

$$\chi_{R_1}(Z) \chi_{R_2}(Z) = \sum_S g_{R_1 R_2 S} \chi_S(Z). \tag{1.2}$$

The Littlewood-Richardson number $g_{R_1 R_2 S}$ counts the number of times the $SU(N)$ irreducible representation S appears in the tensor product $R_1 \otimes R_2$. The product rule follows immediately from the fact that when evaluated on a unitary matrix U , $\chi_R(U)$ gives the character of U in irreducible representation R . These results have been extended to define operators in multimatrix models, that have diagonal two point functions and whose multipoint functions again follow from group theory reasoning [7–9]. For a recent review see [10].

According to the AdS/CFT correspondence [1], these operators have a description in a dual quantum gravity. In particular, for operators with an \mathcal{R} -charge of order N^2 the Schur polynomials should be dual to new geometries [2]. We can now sharpen the question we are asking: the goal of this article is to investigate how this dual geometry emerges. If the Young diagram labeling of operators is to provide a useful tool to explore the AdS/CFT correspondence, the dual geometry should emerge naturally from the Young diagram. We will argue that this is indeed the case. Our approach to this problem entails defining coherent states in the gauge theory that we propose are dual to graviton states. This gives us a way to explore the dual geometry: gravitons move along null geodesics and hence they “know” about the dual geometry. To define the coherent states, we need to define graviton creation and annihilation operators in the Yang-Mills theory. The graviton annihilation and creation operators that we define add or remove boxes to the Young diagram describing the background. The state with a box added/removed needs to be normalized by a factor

of $\sqrt{\frac{c}{N}}$ where c is the weight of the added/removed box. The extra $\sqrt{\frac{c}{N}}$ factor plays a key role in ensuring that the correct dual geometry emerges.

In the next section we start by discussing gravitons in the $\text{AdS}_5 \times \text{S}^5$ background. This serves to illustrate the main ideas in the simplest possible setting. In section 3 we turn to consider LLM geometries; section 4 is concerned with geometries that arise when the field theory background is built from two matrices. In section 5 we discuss our results.

To conclude this introduction, we mention the papers [11, 12]. By probing the $\frac{1}{2}$ BPS operators in the gauge theory with single trace operators that correspond to closed strings these papers have already understood some aspects of the geometry of the dual IIB supergravity. These articles used an eigenvalue density description for the $\frac{1}{2}$ BPS operators in the gauge theory. Using the results of section 3.3 it is easy to see that where our results and those of [11, 12] overlap, they are consistent.

2. Gravitons in $\text{AdS}_5 \times \text{S}^5$

$\mathcal{N} = 4$ super Yang-Mills theory is a conformal field theory. By the state/operator correspondence, the theory on 4 dimensional Euclidean space can be mapped to a theory on $R \times S^3$. Under this map operators map to states and conformal dimensions of operators map into energies of states. We will primarily be interested in the half-BPS operators that can be constructed from the s-waves of complex combinations of the adjoint scalars of the theory on $R \times S^3$. We will make use of two scalars Z, Y ; they have two point function (we suppress the spacetime dependence of this correlator as these spacetime dependences will play no role in this article)

$$\langle Z_{ij} Z_{kl}^\dagger \rangle = \delta_{il} \delta_{jk} = \langle Y_{ij} Y_{kl}^\dagger \rangle.$$

Our computations will mostly be performed in the theory on four dimensional Euclidean space, where we can use the known results for correlators of Schur polynomials and restricted Schur polynomials. To make contact with the dual gravity we will conformally map our results to the theory on $R \times S^3$. The boundary of $\text{AdS}_5 \times \text{S}^5$ in global coordinates is $R \times S^3$ - it is the space that the field theory is defined on; this is why it is easiest to interpret the field theory after mapping to $R \times S^3$. For the operators we consider, after mapping to $R \times S^3$, the dynamics reduces to that of N non-relativistic fermions in an external potential [5, 6, 13]. The phase space of these fermions can be identified with the moduli space of the dual geometries [2].

Chiral primary operators in $\mathcal{N} = 4$ super Yang-Mills theory, with \mathcal{R} charge of $O(1)$ are dual to Kaluza-Klein gravitons on the $\text{AdS}_5 \times \text{S}^5$ background [1]. The \mathcal{R} -charge of the operator maps into the angular momentum (on the S^5) of the graviton. The operator dual to a graviton with one unit of angular momentum is

$$\frac{\text{Tr}(Z)}{\sqrt{N}}.$$

The factor of $N^{-\frac{1}{2}}$ ensures that this operator has a unit two point function and is hence dual to a normalized state. It is straight forward to compute

$$\left\langle \left(\frac{\text{Tr}(Z^\dagger)}{\sqrt{N}} \right)^n \left(\frac{\text{Tr}(Z)}{\sqrt{N}} \right)^m \right\rangle = n! \delta_{m,n}. \quad (2.1)$$

This suggests that the action of $\text{Tr}(Z)/\sqrt{N}$ on the vacuum in the super Yang-Mills theory matches the action of the graviton creation operator in the dual quantum gravity on $\text{AdS}_5 \times S^5$. We will write connections like this using a double sided arrow

$$\frac{\text{Tr}(Z)}{\sqrt{N}} \leftrightarrow a^\dagger,$$

where a^\dagger is a graviton creation operator. It is useful to use the coordinates of [2]. In these coordinates, there is an S^3 extracted from the S^5 . The remaining two coordinates of the S^5 is combined with the radial direction of AdS_5 to make three new coordinates, y, x^1, x^2 . We can also describe the x^1, x^2 plane using an angle ϕ and a radius r . The graviton that is created is an s -wave on the field theory S^3 . It is also an s -wave on the second S^3 (extracted from S^5) and is smeared along the ϕ direction. It is localized at $y = 0$ and at some fixed r . It is this localization that we are trying to describe. A normalized n -graviton state is dual to the operator

$$\frac{1}{\sqrt{n!}} \left(\frac{\text{Tr}(Z)}{\sqrt{N}} \right)^n.$$

We can also define the graviton annihilation operator as

$$\frac{1}{\sqrt{N}} \text{Tr} \left(\frac{d}{dZ} \right) \leftrightarrow a.$$

Note that

$$[a, a^\dagger] = 1 \leftrightarrow \left[\frac{1}{\sqrt{N}} \text{Tr} \left(\frac{d}{dZ} \right), \frac{\text{Tr}(Z)}{\sqrt{N}} \right] = \frac{1}{N} \left(\text{Tr} \frac{d}{dZ} \text{Tr} Z \right) = 1. \quad (2.2)$$

As a check of these identifications, we will now study the dynamics of coherent states of gravitons and verify that we recover the expected graviton dynamics. Define an operator dual to a graviton coherent state as

$$\mathcal{O}_z = \mathcal{N} \exp \left(\frac{\text{Tr}(Z)}{\sqrt{N}} z \right), \quad \left[\frac{1}{\sqrt{N}} \text{Tr} \left(\frac{d}{dZ} \right), \mathcal{O}_z \right] = z \mathcal{O}_z.$$

The normalization \mathcal{N} is easily determined (\bar{z} is the complex conjugate of z)

$$\begin{aligned} \langle \mathcal{O}_z^\dagger \mathcal{O}_z \rangle &= |\mathcal{N}|^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\langle \left(\frac{\text{Tr}(Z)}{\sqrt{N}} \right)^n \left(\frac{\text{Tr}(Z)^\dagger}{\sqrt{N}} \right)^m \right\rangle \frac{z^n \bar{z}^m}{n! m!} \\ &= |\mathcal{N}|^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} \\ &= |\mathcal{N}|^2 e^{|z|^2} \\ &= 1, \end{aligned} \quad (2.3)$$

$$\mathcal{O}_z = e^{-\frac{1}{2}|z|^2} e^{z \frac{\text{Tr}(Z)}{\sqrt{N}}}. \quad (2.4)$$

Since we study $\frac{1}{2}$ BPS operators, the conformal dimension of our operators (which is the graviton's energy in the dual description) equals their \mathcal{R} -charge

$$[\mathcal{R}, \mathcal{O}_z] = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} n \frac{z^n \text{Tr}(Z)^n}{n! \sqrt{N^n}}.$$

After conformally mapping to $R \times S^3$, the operator \mathcal{O}_z is mapped to the coherent state $|z\rangle$, so that ($z = r e^{-i\phi}$)

$$\langle \mathcal{O}_z^\dagger [\mathcal{R}, \mathcal{O}_z] \rangle = \langle z | H | z \rangle = \bar{z} z = r^2, \quad \left\langle \mathcal{O}_z^\dagger i \frac{d}{dt} \mathcal{O}_z \right\rangle = \langle z | i \frac{d}{dt} | z \rangle.$$

The Lagrangian governing the low energy excitations of this coherent state on $R \times S^3$ is

$$L = \langle z | i \frac{d}{dt} | z \rangle - \langle z | H | z \rangle = \dot{\phi} r^2 - r^2.$$

The equations of motion determine $\dot{\phi} = 1$ and $\dot{r} = 0$, which are by now, familiar results for gravitons in $\text{AdS}_5 \times S^5$ [14].

Before moving on we will give a discussion of (2.1). This formula can be obtained using elementary (just Wick contract) methods. We will give a derivation of it using Schur technology since this will generalize nicely in the next section. The operator $\text{Tr}(Z)$ is equal to the Schur polynomial $\chi_\square(Z)$. A product of Schur polynomials is easily computed using the Littlewood-Richardson rule. Using this rule, it is straight forward to verify that (d_R is the dimension of the S_n irreducible representation labeled by R)

$$(\text{Tr}(Z))^n = (\chi_\square(Z))^n = \sum_R d_R \chi_R(Z),$$

and hence that (R is a Young diagram with n boxes; S is a Young diagram with m boxes; the sums run over all possible diagrams with the correct number of boxes)

$$\begin{aligned} \left\langle \left(\frac{\text{Tr}(Z^\dagger)}{\sqrt{N}} \right)^n \left(\frac{\text{Tr}(Z)}{\sqrt{N}} \right)^m \right\rangle &= \left\langle \left(\frac{\chi_\square(Z)^\dagger}{\sqrt{N}} \right)^n \left(\frac{\chi_\square(Z)}{\sqrt{N}} \right)^m \right\rangle \\ &= \delta_{m,n} \sum_R \sum_S \frac{1}{N^n} d_R d_S \langle \chi_R(Z)^\dagger \chi_S(Z) \rangle \\ &= \delta_{m,n} \sum_R (d_R)^2 \frac{f_R}{N^n} \\ &= \delta_{m,n} \sum_R (d_R)^2 \left(1 + O\left(\frac{1}{N^2}\right) \right) \\ &= \delta_{m,n} n! \left(1 + O\left(\frac{1}{N^2}\right) \right), \end{aligned}$$

where we have used the fact that $\sum_R (d_R)^2$ is equal to the order of the group. From our more elementary techniques we know that all corrections to this large N result vanish. Note that this multipoint correlator was computed using the product rule and the two point function for Schur polynomials.

Finally, consider

$$\left\langle \left(\frac{\text{Tr}(Z^{p\dagger})}{\sqrt{pN^p}} \right)^n \left(\frac{\text{Tr}(Z^p)}{\sqrt{pN^p}} \right)^m \right\rangle = n! \delta_{m,n} \left(1 + O\left(\frac{1}{N^2}\right) \right). \quad (2.5)$$

This suggests that

$$\frac{\text{Tr}(Z^p)}{\sqrt{pN^p}}$$

creates gravitons which each carry p units of angular momentum. The result (2.5) can again be obtained by using Schur technology. To do so, we need an expression for $\text{Tr}(Z^p)$, which has been given in [15]. For example, for $p = 2, 3, 4$ we have (in general the right hand side is a sum over all hooks with an alternating sign; the first term corresponding to the symmetric representation is always positive)

$$\begin{aligned} \text{Tr}(Z^2) &= \chi_{\square\square}(Z) - \chi_{\square}(Z), \\ \text{Tr}(Z^3) &= \chi_{\square\square\square}(Z) - \chi_{\square\square}(Z) + \chi_{\square}(Z), \\ \text{Tr}(Z^4) &= \chi_{\square\square\square\square}(Z) - \chi_{\square\square\square}(Z) + \chi_{\square\square}(Z) - \chi_{\square}. \end{aligned} \quad (2.6)$$

3. LLM geometries

In the previous section, we applied $\text{Tr} Z$ and $\text{Tr} \frac{d}{dz}$ to the vacuum of the super Yang-Mills theory to study gravitons in $\text{AdS}_5 \times S^5$. In this section we will study gravitons propagating on different backgrounds. These backgrounds correspond, in the super Yang-Mills theory, to the operator $\chi_B(Z)$ where B is a Young diagram with $O(N^2)$ boxes.

3.1 The annulus

A simple warm up example is provided by the case that B is a Young diagram with M columns and each column contains exactly N boxes; we take M to be $O(N)$ so that $\mu \equiv \frac{M}{N}$ is a number of $O(1)$. Let R be a Young diagram with $n = O(1)$ boxes. We will use the notation $+R$ to denote the Young diagram obtained by stacking R next to B , and the notation $-R$ to denote the Young diagram obtained by removing R from the “bottom right corner” of B as shown in figure 4 below.

Our background is given by taking the “vacuum” to be the normalized state dual to the operator

$$|B\rangle \leftrightarrow \frac{\chi_B(Z)}{\sqrt{f_B}}.$$

Gravitons on this background will correspond to (small) fluctuations about this state.⁴ An operator dual to an n graviton state in the background B is

$$\left(\frac{\text{Tr}(Z)}{\sqrt{N}} \right)^n \frac{\chi_B(Z)}{\sqrt{f_B}} = \left(\frac{\chi_{\square}(Z)}{\sqrt{N}} \right)^n \frac{\chi_B(Z)}{\sqrt{f_B}} = \sum_R d_R \frac{\chi_{+R}(Z)}{\sqrt{f_{BN^n}}}.$$

⁴By small we mean small enough that backreaction can be neglected. This is the case since we take n to be $O(1)$.

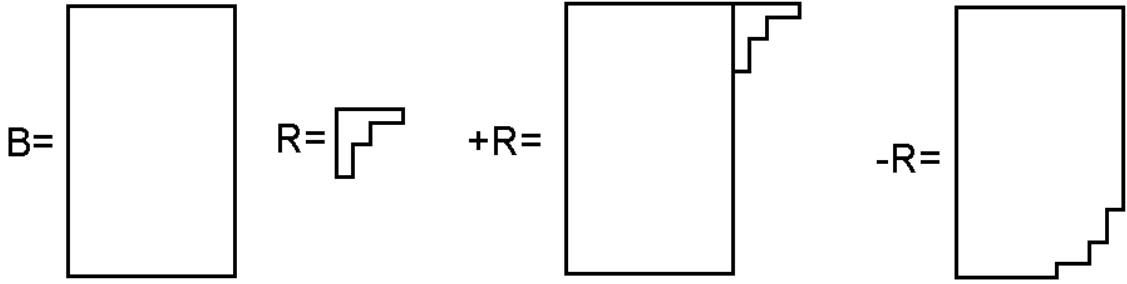


Figure 4: The definition of $+R$ and $-R$ in terms of R and B .

This formula is exact; to get this result, we used the product rule for Schur polynomials. We will now compute the two point correlator of this operator, to the leading order in the large N limit

$$\left\langle \frac{\chi_B(Z)^\dagger}{\sqrt{f_B}} \left(\frac{\text{Tr}(Z)^\dagger}{\sqrt{N}} \right)^n \left(\frac{\text{Tr}(Z)}{\sqrt{N}} \right)^n \frac{\chi_B(Z)}{\sqrt{f_B}} \right\rangle = \sum_R (d_R)^2 \frac{f_{+R}}{f_B N^n} = (1 + \mu)^n n!. \quad (3.1)$$

To see the last equality note that all weights in f_B cancel with weights in f_{+R} , leaving only the weights corresponding to the boxes added to B to produce $+R$; these boxes all have a weight of $N + M + O(1)$ where the $O(1)$ number depends on exactly which box is considered. Given this correlator, a normalized n -graviton state, to leading order in N , is dual to the operator

$$\mathcal{O}_n = \frac{1}{(1 + \mu)^{n/2} \sqrt{n!}} \left(\frac{\chi_{\square}(Z)}{\sqrt{N}} \right)^n \frac{\chi_B(Z)}{\sqrt{f_B}}.$$

Thus, in this new background

$$\frac{\text{Tr}(Z)}{\sqrt{N}} \mathcal{O}_n = \sqrt{\frac{c}{N}} \sqrt{n+1} \mathcal{O}_{n+1} = \sqrt{1 + \mu} \sqrt{n+1} \mathcal{O}_{n+1}, \quad (3.2)$$

where c is the weight of the box added to the background Young diagram B after multiplying by $\text{Tr}(Z) = \chi_{\square}(Z)$. We can write

$$\frac{\text{Tr}(Z)}{\sqrt{N}} \leftrightarrow a^\dagger$$

with a^\dagger a graviton creation operator in the background B . From (3.2) we can read off its action on an n graviton state

$$a^\dagger |n\rangle = \sqrt{\frac{c}{N}} \sqrt{n+1} |n+1\rangle = \sqrt{1 + \mu} \sqrt{n+1} |n+1\rangle.$$

This is perfectly consistent with the trivial background treated in section 2, since in that case, to leading order in N , we have $\frac{c}{N} = 1$. We can consider the action of a derivative on the operator \mathcal{O}_n . It is straight forward to verify that (the rule for the derivative of a Schur polynomial was conjectured in [16] and proved in [17])

$$\left[\frac{\text{Tr} \frac{d}{dZ}}{\sqrt{N}}, \frac{1}{(1 + \mu)^{n/2} \sqrt{n!}} \left(\frac{\chi_{\square}(Z)}{\sqrt{N}} \right)^n \frac{\chi_B(Z)}{\sqrt{f_B}} \right]$$

$$\begin{aligned}
 &= \sqrt{n}\sqrt{1+\mu} \left[\frac{1}{(1+\mu)^{(n-1)/2}\sqrt{(n-1)!}} \left(\frac{\chi_{\square}(Z)}{\sqrt{N}} \right)^{n-1} \frac{\chi_B(Z)}{\sqrt{f_B}} \right] + \dots, \\
 &= \sqrt{n}\sqrt{1+\mu}\mathcal{O}_{n-1} + \dots
 \end{aligned} \tag{3.3}$$

where \dots represents an extra term that does not yet have an obvious interpretation.⁵ Ignoring this term, we can identify,

$$\frac{\text{Tr} \frac{d}{dZ}}{\sqrt{N}} \leftrightarrow a,$$

where the graviton annihilation operator acts as

$$a|n\rangle = \sqrt{\frac{c}{N}}\sqrt{n}|n-1\rangle = \sqrt{1+\mu}\sqrt{n}|n-1\rangle.$$

Clearly $[a, a^\dagger] = 1 + \mu$. However, this can't be correct: the identification

$$\left[\frac{1}{\sqrt{N}} \text{Tr} \left(\frac{d}{dZ} \right), \frac{\text{Tr}(Z)}{\sqrt{N}} \right] \leftrightarrow [a, a^\dagger] = 1 + \mu,$$

implies that

$$\left[\frac{1}{\sqrt{N}} \text{Tr} \left(\frac{d}{dZ} \right), \frac{\text{Tr}(Z)}{\sqrt{N}} \right] = 1 + \mu,$$

which is in stark disagreement with (2.2). To resolve this discrepancy, note that in contrast to the trivial background considered in the previous section, we can now act on the operator dual to the vacuum with the derivative operator to obtain a non-zero result.⁶ Indeed, we find

$$\left(\frac{\text{Tr} \left(\frac{d}{dZ} \right)}{\sqrt{N}} \right)^n \frac{\chi_B(Z)}{\sqrt{f_B}} = \sum_R d_R \prod_i c_i \frac{\chi_{-R}(Z)}{\sqrt{f_B N^n}}.$$

In the last line, $\prod_i c_i$ is the product of the weights of the boxes removed by the derivatives. This formula is again exact. The two point correlator of this operator, to the leading order in the large N limit, is

$$\left\langle \left[\left(\frac{\text{Tr} \left(\frac{d}{dZ} \right)}{\sqrt{N}} \right)^n \frac{\chi_B(Z)}{\sqrt{f_B}} \right]^\dagger \left(\frac{\text{Tr} \left(\frac{d}{dZ} \right)}{\sqrt{N}} \right)^n \frac{\chi_B(Z)}{\sqrt{f_B}} \right\rangle = \mu^n n!.$$

A natural interpretation is that the derivative operator (which lowers the total \mathcal{R} charge of the state) creates gravitons moving in the opposite direction, which we will call *countergravitons*.⁷ An operator dual to a normalized n -countergraviton state, to leading order

⁵This extra term arises from the action of the derivative on $\chi_B(Z)$. The term that we have written explicitly removes some of the boxes that $(a^\dagger)^n$ added, which is the expected action of an annihilation operator.

⁶In the previous section the operator dual to the vacuum was the identity.

⁷We call a state with positive (negative) \mathcal{R} -charge $Q = \text{Tr} Z \frac{d}{dZ}$ after subtracting off the \mathcal{R} -charge of the vacuum a graviton (countergraviton). The term *antigraviton* would be incorrect - antigravitons are created by Z^\dagger ; including any antigravitons in the state would take us out of the $\frac{1}{2}$ BPS sector.

in N , is now given by

$$\mathcal{O}_{-n} = \frac{1}{\mu^{n/2}\sqrt{n!}} \left(\frac{\text{Tr} \left(\frac{d}{dZ} \right)}{\sqrt{N}} \right)^n \frac{\chi_B(Z)}{\sqrt{f_B}}.$$

Denote the countergraviton creation and annihilation operators by b^\dagger and b . In this case

$$\left[\frac{\text{Tr} \left(\frac{d}{dZ} \right)}{\sqrt{N}}, \mathcal{O}_{-n} \right] = \sqrt{\frac{c}{N}} \sqrt{n+1} \mathcal{O}_{-n-1} = \sqrt{\mu} \sqrt{n+1} \mathcal{O}_{-n-1},$$

(where c is the weight of the box removed from the background Young diagram B after acting with $\frac{d}{dZ}$) becomes

$$b^\dagger | -n \rangle = \sqrt{\mu} \sqrt{n+1} | -n-1 \rangle,$$

in the dual quantum gravity. Note that we are using the field theory \mathcal{R} -charge (string theory angular momentum) to label states. Arguing exactly as above, we have $[b, b^\dagger] = \mu$. The resolution to our paradox is now in sight: the derivative operator creates countergravitons and destroys gravitons; multiplication by $\text{Tr}(Z)$ creates gravitons and destroys countergravitons. It is now easy to argue for the identifications⁸

$$\frac{\text{Tr}(Z)}{\sqrt{N}} \leftrightarrow a^\dagger + b, \quad \frac{\text{Tr} \left(\frac{d}{dZ} \right)}{\sqrt{N}} \leftrightarrow a + b^\dagger.$$

These identifications are perfectly consistent with (2.2). Indeed,

$$\left[\frac{1}{\sqrt{N}} \text{Tr} \left(\frac{d}{dZ} \right), \frac{\text{Tr}(Z)}{\sqrt{N}} \right] \leftrightarrow [a + b^\dagger, a^\dagger + b] = (1 + \mu) - (\mu) = 1.$$

Thus, we see that $\text{Tr}(Z)$ and $\text{Tr} \left(\frac{d}{dZ} \right)$ no longer have a local action on the Young diagram B . We will define new operators that are local: split

$$\text{Tr}(Z) = \text{Tr}(Z)_a + \text{Tr}(Z)_b. \tag{3.4}$$

The definition of $\text{Tr}(Z)_a$ is that it can add boxes to B , but only in the upper right hand corner of the Young diagram, corresponding to the action of a^\dagger ; $\text{Tr}(Z)_b$ can add boxes to B , but only in the lower right hand corner of the Young diagram, corresponding to the action of b . Similarly, split

$$\text{Tr} \left(\frac{d}{dZ} \right) = \text{Tr} \left(\frac{d}{dZ} \right)_a + \text{Tr} \left(\frac{d}{dZ} \right)_b.$$

$\text{Tr} \left(\frac{d}{dZ} \right)_a$ removes boxes from the upper right hand corner corresponding to the action of a and $\text{Tr} \left(\frac{d}{dZ} \right)_b$ removes boxes from the lower right hand corner corresponding to the action

⁸These identifications also provide an interpretation for the extra term in equation (3.3): the extra term corresponds to a state with n gravitons and 1 counter graviton.

of b^\dagger . Diagrammatically we have

$$\begin{aligned}
 \text{Tr}(Z)_a \chi_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} &= \chi_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}, & \text{Tr}(Z)_b \chi_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} &= \chi_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}, \\
 \text{Tr}\left(\frac{d}{dZ}\right)_a \chi_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} &= (N+4) \chi_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}, & \text{Tr}\left(\frac{d}{dZ}\right)_b \chi_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} &= (N-1) \chi_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}. & (3.5)
 \end{aligned}$$

We are computing correlation functions using the product rule and known two point functions for Schur polynomials. We have just defined the product of the new local operators with the usual Schur polynomials. This product gives a linear combination of Schur polynomials whose two point function is known. Thus we can compute any correlators using the new local operators which shows that these local operators are indeed well enough defined for our purposes. We will give a more constructive definition of these local operators below.

We can again use the operators we have defined to build a graviton coherent state and check that we reproduce the expected graviton dynamics. The operator dual to a coherent graviton state is

$$\begin{aligned}
 \mathcal{O}_z &= \mathcal{N} \sum_{n=0}^{\infty} \frac{z^n \mathcal{O}_n}{(1+\mu)^{\frac{n}{2}} \sqrt{n!}} \quad \text{where} \\
 \mathcal{O}_n &= \frac{1}{(1+\mu)^{n/2} \sqrt{n!}} \left(\frac{\text{Tr}(Z)_a}{\sqrt{N}} \right)^n \frac{\chi_B(Z)}{\sqrt{f_B}}. & (3.6)
 \end{aligned}$$

Requiring a unit two point function determines the normalization

$$\begin{aligned}
 \langle \mathcal{O}_z^\dagger \mathcal{O}_z \rangle &= |\mathcal{N}|^2 e^{\frac{|z|^2}{1+\mu}} = 1, \\
 \mathcal{O}_z &= e^{\frac{-|z|^2}{2(1+\mu)}} \sum_{n=0}^{\infty} \frac{z^n \mathcal{O}_n}{(1+\mu)^{\frac{n}{2}} \sqrt{n!}}. & (3.7)
 \end{aligned}$$

This operator satisfies

$$\left[\frac{\text{Tr}\left(\frac{d}{dZ}\right)_a}{\sqrt{N}}, \mathcal{O}_z \right] = z \mathcal{O}_z.$$

Denote the coherent state that this operator maps to by $|z\rangle$. The Hamiltonian is

$$H = \frac{a^\dagger a}{1+\mu} - \frac{b^\dagger b}{\mu}. & (3.8)$$

This Hamiltonian measures the energy with respect to the background B . It is for this reason that the second term above is negative: the countergravitons are missing boxes, similar to holes in the Dirac sea. Since the sea is finite the Hamiltonian is bounded below.

To compare results to the dual gravity, we need to translate the background B into an LLM geometry, using the results of [2] which we will now review. This is most easily done by first translating the Schur polynomial into a free fermion state and then translating this

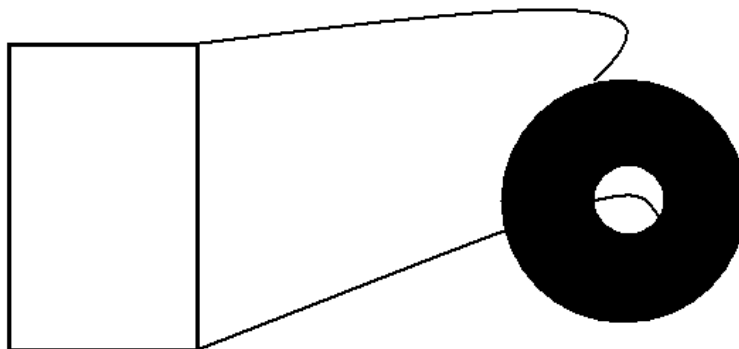


Figure 5: The Young diagram B is shown on the left. The corresponding boundary condition for the LLM geometry is shown on the right.

fermion state into a boundary condition for the LLM geometry. The Schur polynomials map into an N fermion energy eigenstate [5, 6], which is labeled by a set of N integers telling us which levels are occupied. Denote the number of boxes in row i of the Young diagram by r_i ; the row closest to the bottom of the page has $i = 1$ and the row closest to the top of the page has $i = N$. The energies of the occupied levels are given by $r_i + i - 1$. These occupied states map into rings in the free fermion phase space.⁹ To get a correspondence with $\frac{1}{2}$ BPS geometries, identify the free fermion phase space with the LLM plane¹⁰ and color occupied regions in black and unoccupied regions in white. This black and white pattern is the boundary condition for the dual geometry. As an example, we have shown the boundary condition dual to B in figure 5.

By acting with the graviton creation and annihilation operators, we are producing small “ripples” on the inner and outer edges of the annulus. These ripples are localized in the radial direction of the LLM plane in spacetime and they are localized on the Young diagram. We see a direct connection between locality in spacetime and locality on the Young diagram. This is a feature of all of the examples we have considered. We have seen that the oscillators in the new background are scaled by a factor of $\frac{1}{N}$ times the weight of the box removed/added by the oscillator. Does this factor have an interpretation in the dual geometry? The area of the black regions on the LLM plane are proportional to the number of states in the region. We will use a normalization for which the area divided by π/N is exactly equal to the number of states in the black region. Thus, the total area of the black regions is π . The $\text{AdS}_5 \times \text{S}^5$ background itself corresponds to a black disc of radius $R = 1$, in a sea of white. The Young diagram B has a total of N rows and M columns, so that in the free fermion description, there are M unoccupied states followed by N occupied states giving a phase space that looks like an annulus. The hole in the center of the annulus has a total of M states and hence a radius of $\sqrt{\frac{M}{N}} = \sqrt{\mu}$. Note that μ is precisely equal to the weight of the box in the lower right hand corner divided by N . The area of the black

⁹Recall that our fermions are in an external x^2 potential so that fixed energies $E = p^2 + x^2$ are rings in the x, p plane.

¹⁰We call the two dimensional plane on which the LLM boundary condition is specified the LLM plane. This plane is at $y = 0$ and has coordinates x^1, x^2 .

annulus and its white center is proportional to $M + N$ (since together they correspond to $M + N$ states). The outer radius of the annulus is thus $\sqrt{\frac{N+M}{N}} = \sqrt{1+\mu}$. Note that $1+\mu$ is precisely equal to the weight of the box in the upper right hand corner divided by N . Thus, the factor by which the oscillators are rescaled is equal to the value of radius in the LLM plane at which the dual graviton is created.

What is the energy of a state of gravitons created at a given radius on the LLM plane? Given the interpretation in terms of a free fermion phase space, this energy is just the number of gravitons times the energy of a single graviton and the energy of a single graviton is given by the radius squared times N . This tells us that gravitons are small ripples on the outer edge of the annulus and countergravitons are ripples on the inner edge of the annulus.

We will now focus on excitations of the outer edge of the annulus, i.e. those created by a^\dagger . The energy measured by (3.8) is measured with respect to the background B . In the LLM picture, gravitons exciting the outer edge of the annulus have an energy $N(1+\mu)$. This is the energy of the graviton measured with respect to the true vacuum of the theory. There is a natural way to modify (3.8) to get agreement with the LLM result: we can introduce a rescaled time coordinate $\tau = N(1+\mu)t$. In this case, the Hamiltonian becomes

$$H = Na^\dagger a.$$

To see that this rescaling does the job, note that the expectation value of the energy of a coherent state of a^\dagger quanta is

$$\langle z|H|z\rangle = \langle z|a^\dagger a|z\rangle = \bar{z}z = Nr^2.$$

The number operator acts as

$$[N, \mathcal{O}_z] = e^{\frac{-|z|^2}{2(1+\mu)}} \sum_{n=0}^{\infty} \frac{n z^n \mathcal{O}_n}{(1+\mu)^{\frac{n}{2}} \sqrt{n!}},$$

so that the expected number of gravitons, is given by

$$\langle z|N|z\rangle = \langle [N, \mathcal{O}_z] \mathcal{O}_z^\dagger \rangle = \frac{r^2}{1+\mu}.$$

Comparing the expectation value of the energy to the expectation value of the number operator we see that each graviton has an energy of $N(1+\mu)$. The Lagrangian governing the low energy excitations of this coherent state is

$$L = \langle z|i\frac{d}{dt}|z\rangle - \langle z|H|z\rangle = N \left(\frac{d\phi}{d\tau} r^2 - r^2 \right).$$

This is indeed the Lagrangian for the dynamics of gravitons in the LLM geometry (we have reproduced (3.41) of [18]). Notice also that the angular momentum (= \mathcal{R} charge) computed using the rescaled time τ

$$\frac{\partial L}{\partial \frac{d\phi}{d\tau}} = Nr^2$$

is equal to the energy, as expected for a BPS state. A similar argument for a coherent state of countergravitons gives¹¹

$$\langle z|H|z\rangle = \langle z|b^\dagger b|z\rangle = -Nr^2, \quad \langle z|N|z\rangle = \frac{r^2}{\mu}.$$

This suggests that each countergraviton has an energy of $N\mu$. This matches the energy of a graviton exciting the inner edge of the annulus.

It is not surprising that we need to work with the energy measured with respect to the true vacuum, to obtain agreement with the dual gravity picture. The giant graviton probe of [18] is identified with a highly excited fermion; the energy measured with respect to the true vacuum is precisely the energy of this fermion. In addition, the analysis of [21] focuses on very heavy half-BPS states. Almost all such states correspond to Young diagrams whose shapes are small fluctuations around a fixed limiting curve. This characterization of typical states is done using statistical ensembles which assign the energy measured with respect to the true vacuum, to each state. In both cases above the rescaled time coordinate is given by $\tau = ct$ with c the weight of the boxes that are added or removed. Thus, we are lead rather naturally to rescaling time differently for the various sectors of the field theory. The LLM geometries have a time coordinate that is warped, so that there are various “local” times as well as the global time coordinate. Our work might provide a natural way to understand these time rescalings in the field theory.¹² We hope to return to this issue.

Using Schur technology it is simple to verify that

$$\left\langle \frac{\chi_B(Z)^\dagger}{\sqrt{J_B}} \left(\frac{\text{Tr}(Z^{p\dagger})}{\sqrt{pN^p}} \right)^n \left(\frac{\text{Tr}(Z^p)}{\sqrt{pN^p}} \right)^m \frac{\chi_B(Z)}{\sqrt{J_B}} \right\rangle = n![(1 + \mu)^p]^n \delta_{m,n}. \quad (3.9)$$

Thus, we can again identify operators which create gravitons which each carry p units of angular momentum. Does the graviton operator $\text{Tr}(Z^p)$ have a local decomposition? We will now define and give evidence for the *matrix replacement* $Z \rightarrow Z_a + Z_b$. Using the results of [19, 17, 20], we know that we can associate matrices in the Schur polynomial with boxes in the Young diagram. Using this insight, at the level of the Schur polynomial, the replacement amounts to allowing each box to have an identity ‘a’ or ‘b’ and summing over all possible identities. Thus, for example

$$\text{Tr}(Z^2) = \chi_{\square\square}(Z) + \chi_{\square\square}(Z) + \chi_{\square\square}(Z) + \chi_{\square\square}(Z) - \chi_{\square}(Z) - \chi_{\square}(Z) - \chi_{\square}(Z) - \chi_{\square}(Z).$$

When we now take a product of Young diagrams, we use the usual Littlewood-Richardson rule except that we are only allowed to add ‘a’ boxes to the upper right hand region of the background Young diagram and ‘b’ boxes to the lower right hand region of the background Young diagram. This refinement of the Littlewood-Richardson rule defines what we mean by the decomposition $Z \rightarrow Z_a + Z_b$. We have checked that this recipe reproduces (3.9). Thus, for example, $\text{Tr}(Z_a^p)$ creates a localized graviton, localized at the outer edge of the annulus, carrying p units of angular momentum.

¹¹In this case, we have introduced a different rescaled time coordinate $\tau = \mu t$.

¹²We would like to thank the anonymous referee who suggested this interpretation.

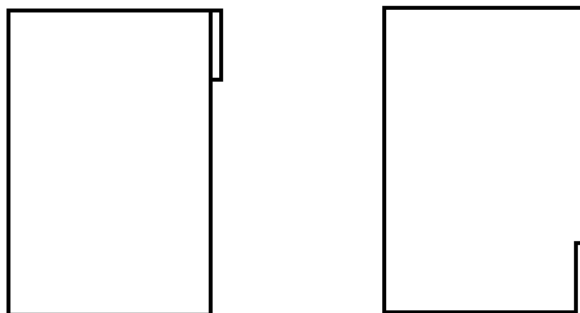


Figure 6: The Young diagrams corresponding to giant gravitons.

Note that, to leading order in N , when B has $O(N^2)$ boxes and $\sum_i p_i$ is $O(1)$, we have (the p_i are all distinct)

$$\begin{aligned}
 & \left\langle \frac{\chi_B(Z)^\dagger}{\sqrt{f_B}} \prod_i \left(\frac{\text{Tr}(Z^{p_i \dagger})}{\sqrt{p_i N^{p_i}}} \right)^n \left(\frac{\text{Tr}(Z^{p_i})}{\sqrt{p_i N^{p_i}}} \right)^m \frac{\chi_B(Z)}{\sqrt{f_B}} \right\rangle \\
 & \equiv \left\langle \prod_i \left(\frac{\text{Tr}(Z^{p_i \dagger})}{\sqrt{p_i N^{p_i}}} \right)^n \left(\frac{\text{Tr}(Z^{p_i})}{\sqrt{p_i N^{p_i}}} \right)^m \right\rangle_B \\
 & = \prod_i \left\langle \left(\frac{\text{Tr}(Z^{p_i \dagger})}{\sqrt{p_i N^{p_i}}} \right)^n \left(\frac{\text{Tr}(Z^{p_i})}{\sqrt{p_i N^{p_i}}} \right)^m \right\rangle_B.
 \end{aligned} \tag{3.10}$$

The fact that the expectation value of the product is the product of expectation values, where we define expectations by

$$\langle O \rangle_B \equiv \left\langle \frac{\chi_B(Z)^\dagger}{\sqrt{f_B}} O \frac{\chi_B(Z)}{\sqrt{f_B}} \right\rangle$$

is in fact a consequence of the fact that $\chi_B(Z)$ creates a new background. Indeed, whenever the expectation value of the product is the product of expectation values, we are dealing with a classical limit.

We have considered excitations that correspond to small ripples at the edges of the annulus. These excitations correspond to point like gravitons and their \mathcal{R} charge (with respect to the background) is $O(1)$. We can also consider excitations with a large \mathcal{R} -charge, of $O(N)$. These excitations correspond to giant gravitons.

To create a giant graviton, we need to build up a single column till it has $O(N)$ boxes stacked in the column; to create a counter-giant graviton we need to remove boxes from a single column till we have a scar of $O(N)$ boxes eroded from the Young diagram. These two possibilities are shown in figure 6. Denote the number of boxes added/eroded to produce the giant by p . Set $\mu_p = \frac{p}{N}$. When we build our coherent state below, we will assume that all giant graviton states contributing to the coherent state have $p + O(1)$ boxes. Thus, the weight of the last box added to produce the giant divided by N is $1 + \mu - \mu_p$ and the weight of the last box eroded to produce the counter-giant divided by N is $\mu + \mu_p$. Using

the methods we introduced above, we can write

$$\frac{\text{Tr}(Z)}{\sqrt{N}} \leftrightarrow a^\dagger + b + A^\dagger + B, \quad \frac{\text{Tr}\left(\frac{d}{dZ}\right)}{\sqrt{N}} \leftrightarrow a + b^\dagger + A + B^\dagger.$$

$a, b, a^\dagger, b^\dagger$ are the operators introduced above which satisfy

$$[a, a^\dagger] = 1 + \mu, \quad [b, b^\dagger] = \mu.$$

The operator A^\dagger adds one unit of energy to the giant; B removes one unit of energy from the counter giant. A and B are not normal bosonic operators - they are Cuntz oscillators, satisfying

$$\begin{aligned} AA^\dagger &= \frac{c}{N} = 1 + \mu - \mu_p, & A^\dagger A &= (1 + \mu - \mu_p)(1 - |0\rangle\langle 0|), \\ BB^\dagger &= \frac{c}{N} = \mu + \mu_p, & B^\dagger B &= (\mu + \mu_p)(1 - |0\rangle\langle 0|). \end{aligned} \quad (3.11)$$

In these formulas, c is again the weight of the box which is added/removed by the oscillator. An operator dual to a normalized giant graviton state with energy n is given by

$$\mathcal{O}_p = \sqrt{\frac{N^p(N+M-p)!}{(N+M)!}} \left(\frac{\text{Tr}(Z)_A}{\sqrt{N}}\right)^p \frac{\chi_B(Z)}{\sqrt{f_B}},$$

an operator dual to a giant graviton coherent state is given by

$$\begin{aligned} \mathcal{O}_z &= \mathcal{N} \sum_{n=0}^{\infty} \frac{z^n}{(1+\mu)_n} \left(\frac{\text{Tr}(Z)_A}{\sqrt{N}}\right)^n \frac{\chi_B(Z)}{\sqrt{f_B}} & (1+\mu)_n &= \prod_{i=1}^n (1+\mu+\mu_i), \\ \mathcal{N} &= \left[\sum_{n=0}^{\infty} \left(\frac{|z|^n}{(1+\mu)_n}\right)^2 \frac{(N+M)!}{(N+M-n)! N^n} \right]^{\frac{1}{2}}. \end{aligned} \quad (3.12)$$

We take z small enough that the expected number of gravitons in this state $p = O(N)$ and $\mu_p \ll 1$. It is straight forward to verify that

$$\left[\frac{\text{Tr}\left(\frac{d}{dZ}\right)}{\sqrt{N}}, \mathcal{O}_p \right] = \sqrt{1+\mu-\mu_p} \mathcal{O}_{p-1}, \quad \left[\frac{\text{Tr}\left(\frac{d}{dZ}\right)}{\sqrt{N}}, \mathcal{O}_z \right] = z \mathcal{O}_z.$$

The Lagrangian governing the low energy excitations of this giant graviton coherent state is¹³

$$L = \langle z | i \frac{d}{dt} | z \rangle - \langle z | H | z \rangle = N \left(r^2 \frac{d\phi}{d\tau} - r^2 \right).$$

which is the correct Lagrangian for D3 branes in an LLM geometry [18].

Finally, consider probing the background B with a string. The same computation, using different methods, has been given in [12]. We probe with a closed string of the form

$$\text{Tr}(Y Z_a^{n_1} Y Z_a^{n_2} Y \dots Y Z_a^{n_L}),$$

¹³ $t = -2N\tau \frac{d}{dr^2} \log \mathcal{N}$; since the equations of motion set r to a constant we have treated r as a constant in performing this rescaling.

corresponding to a string localized at the outer edge of the annulus, or

$$\text{Tr} \left(Y \left(\frac{d}{dZ} \right)_b^{n_1} Y \left(\frac{d}{dZ} \right)_b^{n_2} Y \dots Y \left(\frac{d}{dZ} \right)_b^{n_L} \right),$$

corresponding to a string localized on the inner edge of the annulus. These loops are not $\frac{1}{2}$ -BPS and hence their one loop anomalous dimensions are non-zero. One can define a Cuntz oscillator Hamiltonian whose spectrum gives this one-loop anomalous dimension. The computation of the relevant correlators involving these loops is outlined in appendix B. In the $\text{AdS}_5 \times S^5$ background, these correlators can be computed by associating a lattice with the Y Higgs fields and Cuntz oscillators with the Z_a s and $\left(\frac{d}{dZ}\right)_b$ s. In the background B this is also the case, except that the Cuntz oscillators are rescaled by the (now familiar) square root of the weight divided by N . For Z_a we use

$$\alpha\alpha^\dagger = 1 + \mu, \quad \alpha^\dagger\alpha = (1 + \mu)(1 - |0\rangle\langle 0|),$$

and for $\left(\frac{d}{dZ}\right)_b$ we use

$$\beta\beta^\dagger = \mu, \quad \beta^\dagger\beta = \mu(1 - |0\rangle\langle 0|).$$

The Cuntz oscillator Hamiltonian which gives the one loop anomalous dimension, at large N , is (see appendix B)

$$H = \lambda \sum_{l=1}^L (\alpha_l - \alpha_{l+1})^\dagger (\alpha_l - \alpha_{l+1})$$

for strings localized at the outer edge of the annulus. In the large N limit, where our Cuntz operator representation is valid, the operator

$$\mathcal{O}_z = \sum_{n=0}^{\infty} \frac{z^n}{(1 + \mu)^n} Z_a^n = \sum_{n=0}^{\infty} \frac{z^n}{(1 + \mu)^n} (\alpha^\dagger)^n,$$

is dual to a coherent state. Indeed, it satisfies $[\alpha, \mathcal{O}_z] = z\mathcal{O}_z$. We can put each site of the loop into such a coherent state with a different coherent state parameter; denote the resulting state by $|z_1, z_2, \dots, z_L\rangle$. The semiclassical sigma model action governing the low energy dynamics of this lattice model is given by¹⁴

$$S = \int dt \left(i \langle z_1, z_2, \dots, z_L | \frac{d}{dt} | z_1, z_2, \dots, z_L \rangle - \langle z_1, z_2, \dots, z_L | H | z_1, z_2, \dots, z_L \rangle \right).$$

In the large L limit the above sums become integrals leading to

$$S = L \int dt \int_0^1 d\sigma \left(\frac{(1 + \mu)r^2 \dot{\phi}}{1 + \mu - r^2} - \frac{\lambda}{L^2} \partial_\sigma z \partial_\sigma \bar{z} \right).$$

Comparing this to the fast string limit of the Polyakov action for a string moving in an LLM geometry, we read off (recall that the outer radius of the annulus $R_o = \sqrt{1 + \mu}$)

$$V_\phi(x_1, x_2, y = 0) = \frac{1 + \mu}{1 + \mu - r^2} = \frac{R_o^2}{R_o^2 - r^2},$$

¹⁴ H in this action does not include the leading contribution to the dimension of the operator; it includes only the g_{YM}^2 correction. For the loop we consider the full contribution comes from the F -term. See [11, 12] and references therein.

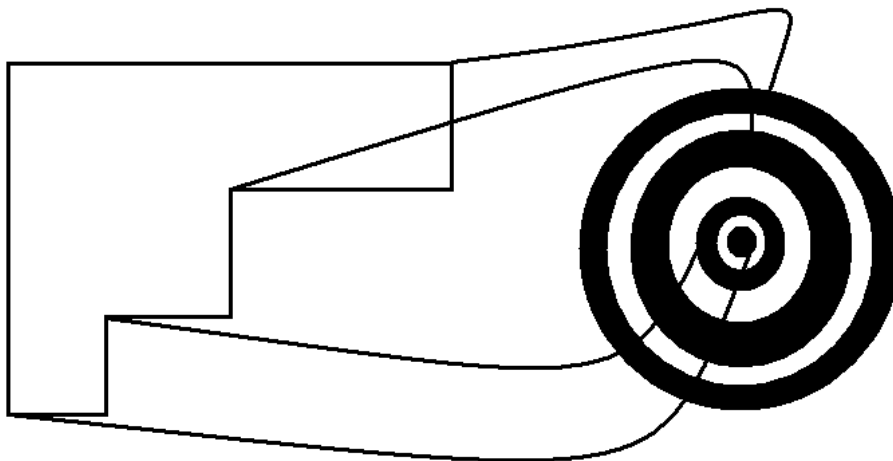


Figure 7: This Young diagram is a generalization of the background considered in the previous section. The outer edge of each ring corresponds to an inward pointing corner.

which is the correct result for $V_\phi(x_1, x_2, y = 0)$ as felt by a string moving at the outer edge of the annulus. Probing the inner edge gives (recall that the inner radius of the annulus $R_i = \sqrt{\mu}$)

$$V_\phi(x_1, x_2, y = 0) = -\frac{\mu}{\mu - r^2} = -\frac{R_i^2}{R_i^2 - r^2}.$$

The negative sign in this formula comes from the fact that when we probe the inner edge we do so with counter excitations. Summing these two gives the complete LLM answer for $V_\phi(x_1, x_2, y = 0)$.

3.2 An arbitrary number of rings

We can generalize the results of the previous section to consider the Young diagram shown in figure 7. The geometry now corresponds to a sequence of concentric rings.

A corner of the Young diagram that we can erode by removing blocks is called an outward pointing corner; a corner of the Young diagram that can be used to grow the diagram by adding blocks is called an inward pointing corner. Fields localized at an outward (inward) pointing corner carry a subscript b (a). The corners identified in figure 7 all correspond to inward pointing corners. $\frac{\text{Tr}(Z)}{\sqrt{N}}$ can be decomposed as

$$\frac{\text{Tr}(Z)}{\sqrt{N}} = \sum_i \frac{\text{Tr}(Z_{a_i})}{\sqrt{N}} + \sum_j \frac{\text{Tr}(Z_{b_j})}{\sqrt{N}},$$

where i is summed over all inward pointing corners and j is summed over all outward pointing corners. Each of the local fields on the right hand side can only add boxes at their corner. $\frac{\text{Tr}(Z)}{\sqrt{N}}$ corresponds, in the dual quantum gravity, to a sum of oscillator $\sum_i a_i^\dagger + \sum_j b_j$. Similarly,

$$\frac{1}{\sqrt{N}} \text{Tr} \left(\frac{d}{dZ} \right) = \sum_i \frac{1}{\sqrt{N}} \text{Tr} \left(\frac{d}{dZ_{a_i}} \right) + \sum_j \frac{1}{\sqrt{N}} \text{Tr} \left(\frac{d}{dZ_{b_j}} \right)$$

corresponds to the sum of oscillator $\sum_i b_i^\dagger + \sum_j a_j$ where again i is summed over all outward pointing corners and j is summed over all inward pointing corners. It is a simple matter to check that

$$[a_i, a_i^\dagger] = \frac{c_{a_i}}{N},$$

where c_{a_i} is the weight of the boxes at the i th inward pointing corner and

$$[b_j, b_j^\dagger] = \frac{c_{b_j}}{N},$$

where c_{b_j} is the weight of the boxes at the j th outward pointing corner. Finally, arbitrary mixed correlators, for example of the form

$$\begin{aligned} \left\langle \frac{\chi_B(Z)}{\sqrt{f_B}} \text{Tr} \left((Z_{a_i})^n (Z_{a_i}^\dagger)^n \right) \frac{\chi_B(Z)^\dagger}{\sqrt{f_B}} \right\rangle &\equiv \left\langle \text{Tr} \left((Z_{a_i})^n (Z_{a_i}^\dagger)^n \right) \right\rangle_B \\ \left\langle \left[\left(\frac{d}{dZ_{b_i}} \right)^n \right]_{ij} \frac{\chi_B(Z)}{\sqrt{f_B}} \left[\left(\frac{d}{dZ_{b_i}^\dagger} \right)^n \right]_{ji} \frac{\chi_B(Z)^\dagger}{\sqrt{f_B}} \right\rangle &\equiv \left\langle \text{Tr} \left(\left(\frac{d}{dZ_{b_i}} \right)^n \left(\frac{d}{dZ_{b_i}^\dagger} \right)^n \right) \right\rangle_B \end{aligned} \quad (3.13)$$

are easily computed using the modified ribbon graphs

$$\begin{aligned} \left\langle (Z_{a_i})_{ij} (Z_{a_i}^\dagger)_{kl} \right\rangle_B &= \frac{c_{a_i}}{N} \delta_{il} \delta_{jk}, \\ \left\langle \left(\frac{d}{dZ_{b_j}} \right)_{ij} \left(\frac{d}{dZ_{b_j}^\dagger} \right)_{kl} \right\rangle_B &= \frac{c_{b_j}}{N} \delta_{il} \delta_{jk}. \end{aligned} \quad (3.14)$$

3.3 Eigenvalues

The split of both $\text{Tr}(Z)$ and $\text{Tr}\left(\frac{d}{dZ}\right)$ into local operators has been defined by stating what action these operators have on the background Young diagram. Is there a more direct definition which would allow us to construct these local (in the bulk) operators? The $\frac{1}{2}$ BPS sector of the theory on $R \times S^3$ is governed by the singlet sector dynamics of the action

$$S = \int dt \text{Tr} \left(\frac{1}{2} \frac{dZ}{dt} \frac{dZ^\dagger}{dt} - \frac{1}{2} Z Z^\dagger \right).$$

The eigenvalue dynamics of this model is the dynamics of N nonrelativistic fermions [5, 6, 21, 22]. The eigenvalues of Z are related to the positions of the fermions. Now, recall a few facts from the usual harmonic oscillator: the position $X \sim a^\dagger + a$ has a vanishing expectation value for any energy eigenstate; for X^2 however we find

$$\langle n | (a + a^\dagger)^2 | n \rangle = \langle n | (aa^\dagger + a^\dagger a) | n \rangle \sim n,$$

so that the expectation value of X^2 is large if n is large.

In the large N limit, we expect that the eigenvalues of Z can be described by a classical configuration. Armed with our harmonic oscillator insight, we expect that the eigenvalues will be large if the corresponding fermion is highly excited. A highly excited fermion corresponds to a long row in the Young diagram. Thus, the Young diagram of the background B is a ‘‘picture’’ of the eigenvalues of the classical large N configuration

for Z in the sense that there is an eigenvalue for each row in the Young diagram and the size of the eigenvalue is related to the length of the row. For example, the background corresponding to the Young diagram pictured in figure 7 will have eigenvalues clumped around four values. Z can thus be split into four submatrices which each have eigenvalues of roughly the same size. We suggest that this split is precisely the split of Z into local operators.

It is easy to gather numerical support for this proposal. Consider a matrix

$$Z = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

The product

$$\chi_{\square\square}(Z)\chi_{\square}(Z) = \chi_{\square\square\square}(Z) + \chi_{\square\square\square\square}(Z)$$

follows as a simple application of the Littlewood-Richardson rule. Our local operators will satisfy

$$\chi_{\square\square}(Z)\chi_{\square}(Z) = \chi_{\square\square\square}(Z), \quad \chi_{\square\square}(Z)\chi_{\square}(Z) = \chi_{\square\square\square}(Z).$$

Consider the case that $\lambda_1 \gg \lambda_2$. Then our proposal says $Z = Z_a + Z_b$ with

$$Z_a = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Z_b = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

We thus have the predictions

$$\chi_{\square\square}(Z)\lambda_1 = \chi_{\square\square\square}(Z), \quad \chi_{\square\square}(Z)\lambda_2 = \chi_{\square\square\square}(Z),$$

which should be true when $\lambda_1 \gg \lambda_2$.

In figure 8 we have plotted

$$F = \frac{\chi_{\square\square\square}(Z)\lambda_2}{\chi_{\square\square\square}(Z)},$$

against $y = \frac{\lambda_1}{\lambda_2}$. As y increases, F very rapidly approaches 1, confirming our proposal.

Thus, locality on the Young diagram maps, at least for widely separated eigenvalues, into locality on the eigenvalues.

4. Beyond LLM

The LLM geometries correspond, in the super Yang-Mills theory, to operators built using a single matrix Z . In this section we would like to consider geometries which correspond to operators built using two matrices X and Z . These backgrounds will include the $\frac{1}{4}$ BPS geometries of [23]; we are not yet able to make contact with these geometries. The operators we have in mind are the restricted Schur polynomials of [9]

$$\chi_{R,(r_n,r_m)}(Z,X) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \text{Tr}_{(r_n,r_m)}(\Gamma_R(\sigma)) Z_{i_{\sigma(1)}}^{i_1} \cdots Z_{i_{\sigma(n)}}^{i_n} X_{i_{\sigma(n+1)}}^{i_{n+1}} \cdots X_{i_{\sigma(n+m)}}^{i_{n+m}}. \tag{4.1}$$

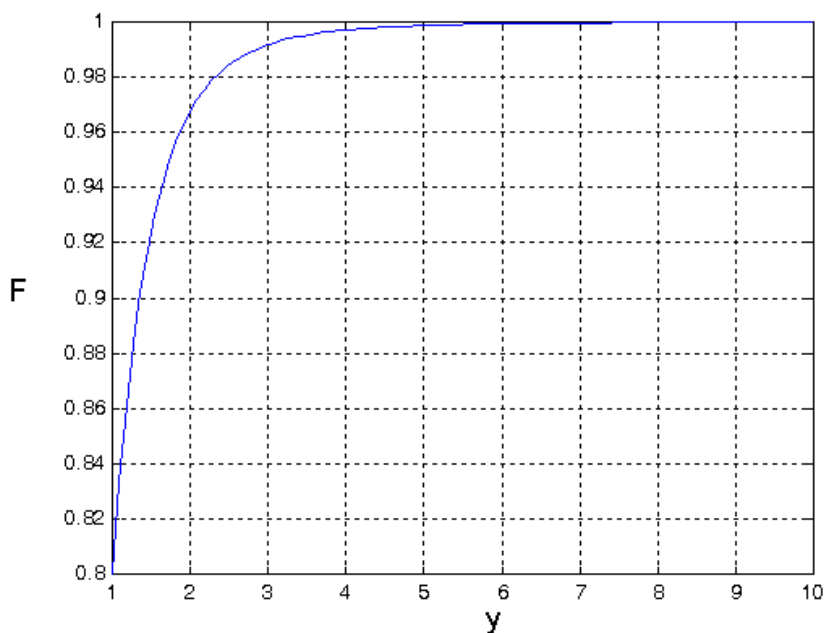


Figure 8: A plot of F against $y = \lambda_1/\lambda_2$.

The two point function of these operators is trivial to compute using the results of [9]. They also satisfy a product rule as shown in [24]. Thus we can compute the correlators needed to probe the geometry with gravitons. The precise form of the product rule for restricted Schur polynomials involves the computation of restricted traces; we do not yet have efficient methods for this. However, in the special case that r_n is a Young diagram with M columns and each column contains exactly N boxes we can write [24]

$$\chi_{+r_m, (r_n, r_m)}(Z, X) = \frac{(\text{hooks})_{r_n} (\text{hooks})_{r_m}}{(\text{hooks})_{+r_m}} \chi_{r_n}(Z) \chi_{r_m}(X).$$

After normalizing this becomes (the hat denotes a normalized operator)

$$\hat{\chi}_{+r_m, (r_n, r_m)}(Z, X) = \frac{\chi_{r_n}(Z)}{\sqrt{f_{r_n}}} \frac{\chi_{r_m}(X)}{\sqrt{f_{r_m}}} \equiv \chi_B(Z, X).$$

Clearly, in this case the required product rule is just the Littlewood-Richardson rule. Thus, we can study these backgrounds using the methods of section 3. We will not be so general: in what follows we will assume that r_n and r_m are both rectangles that have N rows and that r_n has M_n columns and r_m M_m columns. We take both M_n and M_m to be $O(N)$ so that $\mu_n \equiv \frac{M_n}{N}$ and $\mu_m \equiv \frac{M_m}{N}$ are numbers of $O(1)$. The backgrounds we consider are perhaps the most trivial that could be considered. Indeed, all traces that appear contain only X s or Z s. It would be much more interesting to consider backgrounds which are dual to operators that have traces over a product of X s and Z s.

It is straight forward to verify that

$$\begin{aligned} \mathcal{O}_p &= \frac{1}{(1 + \mu_n)^{p/2} \sqrt{p!}} \left(\frac{\chi_{\square}(Z)}{\sqrt{N}} \right)^p \chi_B(Z, X), \\ \mathcal{O}_p &= \frac{1}{(1 + \mu_m)^{p/2} \sqrt{p!}} \left(\frac{\chi_{\square}(X)}{\sqrt{N}} \right)^p \chi_B(Z, X), \end{aligned} \tag{4.2}$$

are normalized p graviton states, as are

$$\begin{aligned} \mathcal{O}_{-p} &= \frac{1}{\mu_n^{p/2} \sqrt{p!}} \left(\frac{\text{Tr} \left(\frac{d}{dZ} \right)}{\sqrt{N}} \right)^p \chi_B(Z, X), \\ \mathcal{O}_{-p} &= \frac{1}{\mu_m^{p/2} \sqrt{p!}} \left(\frac{\text{Tr} \left(\frac{d}{dX} \right)}{\sqrt{N}} \right)^p \chi_B(Z, X). \end{aligned}$$

Thus, even in this more general two matrix background we can define operators dual to local gravitons. $\text{Tr}(X)$ and $\text{Tr}(Z)$ create these gravitons and the state must again be normalized by the factor c/N that appeared above; this suggests that these gravitons propagate on some classical geometry. For the gravitons created by $\text{Tr}(Z)$ c is read from r_n ; for the gravitons created by $\text{Tr}(X)$ c is read from r_m . This result is easy to interpret: the gravitons built using Z s feel only the background of $\chi_{r_n}(Z)$ whilst the gravitons built using X s feel only the background of $\chi_{r_m}(X)$. This is a consequence of the fact that our background takes the form of a Schur polynomial in X times a Schur polynomial in Z . For more general backgrounds we do not expect that this will be the case. This result suggests that it may be possible to construct some nontrivial $\frac{1}{4}$ BPS geometries by cleverly combining LLM geometries.

5. Discussion

The Schur polynomials (and their multimatrix generalizations) provide a convenient way to organize the interactions of a matrix model - by using the Schur polynomials we have been able to compute correlators that involve $O(N^2)$ fields. Direct confrontation of these correlators using standard Feynman diagrams is daunting; using the Schur polynomials, these computations are simple. The Schur polynomials have two properties that are ultimately responsible for this simplicity: (i) one can compute their two point function exactly and further they diagonalize the two point function; (ii) they satisfy a simple product rule which allows a straight forward computation of n -point correlators.

The Schur polynomials are a nontrivial linear combination of terms which each have a different trace structure - the number of traces and the number of fields in each trace vary from term to term. The Young diagram is a book keeping device which keeps track of this structure i.e. it summarizes exactly how the Schur polynomial is constructed.

We have demonstrated some features of how the *local* geometry dual to operators with a large \mathcal{R} -charge emerges from the super Yang-Mills theory. In particular, we have described how to define super Yang-Mills operators that are dual to localized excitations in the bulk of the dual quantum gravity. The Young diagram labeling of the operators

has played a crucial role. A Young diagram can be translated into an LLM boundary condition - it corresponds to a set of concentric rings. Any LLM boundary condition that is a set of concentric rings can be translated into a Young diagram. Localized excitations in spacetime are also localized on the Young diagram. An excitation at a specific radius in the LLM plane has an energy equal to the radius squared; thus by constructing excitations of a specific energy we have constructed excitations localized at a specific radius and by measuring the energy of the excitation we can determine where the excitation is localized. We have defined local creation and annihilation operators that add or remove boxes at a specific location on the Young diagram. These operators are normalized with a factor equal to $\frac{c}{N}$ where c is the weight of the box which was added/removed. The excitations that we have studied are confined to move on the $y = 0$ LLM plane and orbit with $\dot{\phi} = 1$ at a fixed radius r_o . This normalization factor $\frac{c}{N}$ where c is the weight of the box corresponding to the localized excitation, is also equal to the radius squared r_o^2 . Of course, as we have reviewed, the weights of the Young diagram encode the specific way in which the indices of the matrices making up the operator were symmetrized; we see that they also tell us the radii of the rings of the LLM boundary condition.

The Hamiltonian obtained from the field theory measures energy with respect to the background. By introducing a rescaled time coordinate $\tau = ct$, we can ensure that the Hamiltonian measures energy with respect to the true vacuum of the theory. In this way, we are lead rather naturally to rescaling time differently for the various sectors of the field theory. The LLM geometries have a time coordinate that is warped, so that there are various “local” times as well as the global time coordinate. Our work might provide a natural way to understand these time rescalings in the field theory.

A standard way to describe the large N limit of a matrix model is in terms of a density of eigenvalues. A particular background would correspond to a specific eigenvalue density. In this article we have used a Young diagram to specify the background. We have seen however, that this Young diagram can be translated into a particular eigenvalue distribution. A Young diagram describing m rings will correspond to a distribution in which the eigenvalues are clustered into m “clumps” with eigenvalues in a particular clump having similar magnitudes. This suggests that the free fermion picture of the eigenvalue dynamics has a potential with m minima. The locality that we have seen emerge from the field theory emerges when the clumps of eigenvalues have well separated values. This matches nicely with other recent results [25].

Finally, note that the reorganization of the degrees of freedom that we have discussed in this article has a natural extension to multimatrix models. Indeed, the multimatrix operators given in [9] have a diagonal two point function and satisfy a generalization of the Littlewood-Richardson rule [24]. More work is however needed to write the product rule satisfied by the restricted Schur polynomials in a useful form. See also [7, 8] for alternative multimatrix operators that can also be analyzed by exploiting group theory techniques.

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A. A review of the LLM geometries

The LLM geometries [2] are regular $\frac{1}{2}$ BPS solutions of type IIB string theory that are asymptotically $\text{AdS}_5 \times \text{S}^5$. They have an $R \times \text{SO}(4) \times \text{SO}(4)$ isometry group. The metric is given by

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2,$$

where

$$\begin{aligned} h^{-2} &= 2y \cosh G, & z &= \frac{1}{2} \tanh G, \\ y\partial_y V_i &= \epsilon_{ij} \partial_j z, & y(\partial_i V_j - \partial_j V_i) &= \epsilon_{ij} \partial_y z. \end{aligned} \quad (\text{A.1})$$

The function z is obtained by solving

$$\partial_i \partial_i z + y \partial_y \frac{\partial_y z}{y} = 0.$$

Regularity requires that $z = \pm \frac{1}{2}$ on the $y = 0$ plane. We will often trade the x^1, x^2 coordinates of the $y = 0$ plane for an angle and a radius r, ϕ or for the complex coordinates $z = x^1 + ix^2, \bar{z} = x^1 - ix^2$.

For a set of rings having a total of E edges with radii R_l $l = 1, 2, \dots, E$ we have

$$\begin{aligned} z &= \sum_{l=1}^E \frac{(-1)^{E-l}}{2} \left(\frac{r^2 + y^2 - R_l^2}{\sqrt{(r^2 + y^2 + R_l^2)^2 - 4r^2 R_l^2}} - 1 \right), \\ V_\phi(x^1, x^2, y) &= \sum_{l=1}^E \frac{(-1)^{E-l+1}}{2} \left(\frac{r^2 + y^2 + R_l^2}{\sqrt{(r^2 + y^2 + R_l^2)^2 - 4r^2 R_l^2}} - 1 \right). \end{aligned} \quad (\text{A.2})$$

The fast string limit [26] of the Polyakov action for a string moving in an LLM geometry is given by [11, 12]

$$S = L \int d\tau \int_0^1 d\sigma \left(\frac{i}{2} V \dot{z} - \frac{i}{2} \bar{V} \dot{\bar{z}} - \frac{\lambda}{L^2} \partial_\sigma z \partial_\sigma \bar{z} \right).$$

B. Correlators

In this section we will compute the correlators needed to probe a general background B with a closed string. In terms of the loop

$$\mathcal{O} = \text{Tr} (Y Z_a^{n_1} Y Z_a^{n_2} Y \dots Y Z_a^{n_L}),$$

we wish to compute

$$\langle \mathcal{O}\mathcal{O}^\dagger \rangle_B \equiv \left\langle \frac{\chi_B}{\sqrt{f_B}} \mathcal{O}\mathcal{O}^\dagger \frac{\chi_B^\dagger}{\sqrt{f_B}} \right\rangle.$$

Assume the n_i are all distinct. The leading contribution at large N comes from the following contraction of the Y s

$$\langle \mathcal{O}\mathcal{O}^\dagger \rangle_B \equiv \left\langle \frac{\chi_B}{\sqrt{f_B}} \prod_{i=1}^L \text{Tr} (Z_a^{n_i} (Z_a^\dagger)^{n_i}) \frac{\chi_B^\dagger}{\sqrt{f_B}} \right\rangle.$$

Look back at the correlation functions (2.1) and (2.5). These correlators are easy to reproduce using the two point function

$$\langle Z_{ij} Z_{kl}^\dagger \rangle = \delta_{il} \delta_{jk}.$$

Now consider (3.1) and (3.9). These correlators are clearly reproduced if we take

$$\langle (Z_a)_{ij} (Z_a^\dagger)_{kl} \rangle = (1 + \mu) \delta_{il} \delta_{jk}.$$

By considering more general correlators, it is straight forward to see that all such correlators are reproduced by the rule

$$\langle (Z_{\text{loc}})_{ij} (Z_{\text{loc}}^\dagger)_{kl} \rangle = \frac{c}{N} \delta_{il} \delta_{jk}, \tag{B.1}$$

where Z_{loc} acts locally on the Young diagram, adding boxes with weight c up to addition of an irrelevant (at large N) $O(1)$ number. This clearly reproduces the correlators computed using Schur technology.

Focus again on the case that B is the annulus. With the result (B.1) in hand we find

$$\langle \mathcal{O}\mathcal{O}^\dagger \rangle_B = \left\langle \frac{\chi_B}{\sqrt{f_B}} \prod_{i=1}^L \text{Tr} (Z_a^{n_i} (Z_a^\dagger)^{n_i}) \frac{\chi_B^\dagger}{\sqrt{f_B}} \right\rangle = \prod_{i=1}^L N^{n_i+1} (1 + \mu)^{n_i}.$$

Clearly it is now straight forward to compute correlators needed when probing the background with a closed string state.

A very convenient way to account for the planar contractions is by introducing Cuntz oscillators [28]. The matrix two point function determines the Cuntz oscillator algebra. In terms of Cuntz oscillators (B.1) becomes

$$\alpha\alpha^\dagger = \frac{c}{N}, \quad \alpha^\dagger\alpha = \frac{c}{N}(1 - |0\rangle\langle 0|).$$

The analysis of [27] can now be applied to obtain the Cuntz Hamiltonian quoted in section 3.1.

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